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NATURAL-CONVECTION ON A FINITE-SIZE HORIZONTAL PLATE*

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Abstract—The problem of two-dimensional, steady-state, natural-convection on a finite-size, isothermal, horizontal plate is examined theoretically for the case in which a cold plate faces upwards or a hot plate faces downwards. The boundary layer equations of continuity, energy and momentum are solved using an integral analysis in order to determine the heat transfer at the plate surface. An essential part of this analysis is the use of the concept or condition that the boundary layer depth at the plate edge is equal to a critical depth. The results of this analysis agree very well with the existing experimental correlation equation for air.

NOMENCLATURE

b, plate width;
g, acceleration of gravity;
Gr, Grashof number, $L^3 g\beta\Delta T_w/\nu^2$;
k, thermal conductivity;
L, plate half length;
$\dot{M}$, mass flow rate;
Nu, Nusselt number, hL/k;
p, pressure;
p_0 pressure of surroundings;
Pr, Prandtl number, $c_p\mu/k$;
$\dot{Q}$, heat transfer;
t, dimensionless boundary layer depth;
T, temperature;
T_0, temperature of surroundings;
T_w, wall or plate temperature;
u, v, horizontal and vertical velocity components;
x, y, horizontal and vertical coordinate axes;
$\bar{x}$, dimensionless coordinate;
y_0, reference height;
α, thermal diffusivity;
β, elliptic integral argument, or coefficient of thermal expansion;
δ, boundary layer depth;
δ_c, critical depth at the plate's edge;
δ_0, depth at the plate center;
$\bar{\delta}$, dimensionless boundary layer depth, equation (11);
μ, viscosity;
ν, kinematic viscosity;
ρ, density;
ρ_0, density of surroundings.

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INTRODUCTION

A THEORETICAL investigation of the two-dimensional, steady-state, free-convection boundary layer on a finite-size, isothermal, horizontal plate is described in this paper. More specifically, the problem of interest is that of either a cold plate facing upwards or a hot plate facing downwards; the same physical phenomena occurs in both cases. For convenience, a cold plate facing upwards will be considered (see Fig. 1). Referring to Fig. 1, as the fluid near the surface is cooled by the cold wall it becomes more dense than the surrounding fluid and tends to accumulate on the plate's surface. This denser fluid flows across the plate, under the influence of a hydrostatic pressure gradient, and off the edges. Thus, under steady-state conditions a natural convection boundary layer is established that has a maximum depth at the plate's center and decreases to a critical depth (see Appendix A) at the edges. The mass flow rate within the boundary layer increases from zero at $x = 0$ to a maximum value at the edge of the plate. Figure 1 shows typical velocity distributions through the boundary layer, $u(y)$, at different positions along the plate's surface.

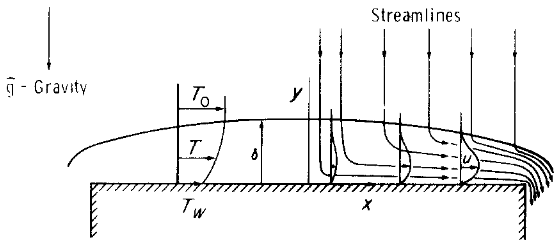

FIG. 1. Natural convection boundary layer on a finite-size horizontal plate.

This problem has been studied experimentally by Weise [1] and Griffiths and Davis [2] and theoretically by Suriano and Yang [3]. Reference [4] gives a correlation of the existing experimental data. Suriano and Yang used a numerical (finite-difference) method to solve the equations describing the two-dimensional, free-convection flow field in the vicinity of a finite-size, horizontal plate for the case of relatively small Rayleigh numbers.

The purpose of this paper is to describe the behaviour of the free-convection boundary layer on a finite-size, horizontal, cold plate facing upward (or a hot plate facing downward) for the case of large Rayleigh or Grashof numbers.

An investigation [5] of the possibility of obtaining a similar solution to the equations which describe this free convection boundary layer has shown that a similarity transformation cannot be found by current methods [6, 7] which will satisfy the boundary conditions of the problem. The present paper, therefore, describes an integral analysis of the equations of continuity, momentum and energy, which describe this particular problem. A very important part of this analysis is the use of the concept or condition that a critical boundary-layer depth exists at the plate's edges (see Appendix A).

ANALYSIS

It is shown in Appendix B that, for large values of Gr and $GrPr^2$ and corresponding small values of $(\delta/L)^2$, the two-dimensional, steady-state, free-convection, boundary layer on a horizontal plate can be described by the following equations:

x-momentum

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad \text{(1a)}$$

y-momentum

$$\frac{\partial p}{\partial y} = -\rho_0[1 - \beta(T - T_0)]g, \quad \text{(1b)}$$

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{(1c)}$$

energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \quad \text{(1d)}$$

The order-of-magnitude analysis in Appendix B shows that equations (1) adequately describe the boundary-layer behavior when $(\delta/L)^2 < 0{\cdot}1$ which corresponds to conditions of Gr and $GrPr^2 > 10^5$. The x- and y-momentum equations can be combined to yield

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\int_y^\infty \frac{\partial}{\partial x}(T - T_0)\,dy + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{(2)}$$

if the density variation is considered to affect only the bouyancy term. The boundary conditions are

at $y = 0$, $u = v = 0$, and $T = T_w$ = const..

$$\text{as } y \to \infty, u \to 0, T \to T_0, \frac{\partial u}{\partial y} \to 0, \frac{\partial T}{\partial y} \to 0,$$

$$\text{and at } x = 0, u(y) = 0. \tag{3}$$

If the continuity equation (1c) is combined with the energy (1d) and the momentum (2) equations and the results integrated with respect to y from $y = 0$ to $y = \delta$, the following relations are obtained:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u^2\,\mathrm{d}y = -\nu\left(\frac{\partial u}{\partial y}\right)_w + g\beta\int_0^\delta\left[\int_y^\delta \frac{\partial}{\partial x}(T - T_0)\,\mathrm{d}y\right]\mathrm{d}y, \tag{4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u(T - T_0)\,\mathrm{d}y = -\alpha\left(\frac{\partial T}{\partial y}\right)_w \tag{4b}$$

Equations (4a) and (4b) represent the integral form of the momentum and energy equations, including the continuity equation, which describe the free convection boundary layer on a horizontal plate.

Suitable approximate expressions [8] for the velocity, u, and temperature, T, distributions within the free-convection boundary layer and which satisfy the boundary conditions given by equations (3) are

$$u = U\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2, \tag{5a}$$

$$\frac{T - T_o}{T_w - T_o} = \frac{T - T_o}{\Delta T_w} = \left(1 - \frac{y}{\delta}\right)^2, \tag{5b}$$

where U is a function of x which has the dimensions of velocity and is to be determined. With the above expressions for u and T, equations (4a) and (4b) become

$$\frac{1}{105}\frac{\mathrm{d}}{\mathrm{d}x}(U^2\delta) = \frac{-\nu U}{\delta} + \frac{g\beta\Delta T_w}{6}\delta\frac{\mathrm{d}\delta}{\mathrm{d}x} \tag{6a}$$

$$\delta\frac{\mathrm{d}}{\mathrm{d}x}(U\delta) = 60\alpha. \tag{6b}$$

Equations (6) are two first order, nonlinear differential equations which must be solved simultaneously. A particular or physically reasonable solution of these equations, therefore, requires two boundary conditions, one of which corresponds to the condition of symmetry at $x = 0$, i.e.

$$U(0) = 0, \quad \text{or} \quad \frac{\mathrm{d}\delta}{\mathrm{d}x}(0) = 0. \tag{7}$$

A slight rearrangement of equations (6) will show that $U = 0$ when $\mathrm{d}\delta/\mathrm{d}x = 0$; these are the physical conditions which must exist at the plate center.

The second boundary condition involves the boundary layer depth at the plate's edge. This depth was established by application of a minimum mechanical energy principle [5, 9], such as that used in open-channel hydraulics. The principle states that a fluid flowing across a horizontal plate under the influence of a hydrostatic pressure gradient and off the plate's edge will adjust itself so that the flow rate of mechanical energy within the fluid will be a minimum with respect to the boundary-layer depth at the plate's edge. Although this principle has in the past been applied to flows with a free surface, its application to the horizontal plate, free-convection, boundary layer is a plausible extension not inconsistent with the boundary layer assumption itself. The application of the above principle (see Appendix A) yields the following result for the boundary-layer depth, to be called the *critical depth*, δ_c, at the plate's edge:

$$\delta_c = \delta(L) = \left(\frac{108\dot{M}_c^2}{5\rho^2 b^2 g\beta\Delta T_w}\right)^{\frac{1}{3}}. \tag{8}$$

Equation (8) can be expressed in a more convenient form if the mass flow rate, $\dot{M}$, is expressed as

$$\dot{M} = \int_0^\delta \rho u b\,\mathrm{d}y = \frac{\rho U b\delta}{12} = 5\rho b\alpha\int_0^x \frac{\mathrm{d}x}{\delta}, \tag{9}$$

where equations (5a) and (6b) have been used. By combining equations (8) and (9) the condition at the plate's edge becomes

$$\delta_c = \left[\frac{540\alpha^2\left(\int\limits_0^L \frac{\mathrm{d}x}{\delta}\right)^2}{\beta g \Delta T_w}\right]^{1/4}. \qquad (10)$$

If nondimensional variables, $\bar{\delta}$, $\bar{U}$ and $\bar{x}$, are defined as

$$\bar{\delta} = \delta/J^{1/4}, \qquad \bar{x} = x/L,$$
$$\bar{U} = U/[J^{1/4} g\beta\Delta T_w/(6\nu L)],$$

where

$$J = 1440.\, \nu^2 L^2/(g\beta\Delta T_w Pr).$$

Equations (6) and boundary conditions (7) and (10) can be expressed as

$$\frac{16}{7Pr}\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(\bar{U}^2\bar{\delta}) = -\frac{\bar{U}}{\bar{\delta}} - \bar{\delta}\frac{\mathrm{d}\bar{\delta}}{\mathrm{d}\bar{x}}, \qquad (11a)$$

$$4\bar{\delta}\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(\bar{U}\bar{\delta}) = 1 \qquad (11b)$$

$$\bar{U}(0) = 0, \quad \text{or} \quad \frac{\mathrm{d}\bar{\delta}}{\mathrm{d}\bar{x}}(0) = 0, \quad \text{and} \qquad (11c)$$

$$\bar{\delta}(L) = \bar{\delta}_c = \left[\frac{3}{8Pr}\left(\int\limits_0^1 \frac{\mathrm{d}\bar{x}}{\bar{\delta}}\right)^2\right]^{1/4}. \qquad (11d)$$

The following sections describe solutions of equations (11) for two different conditions. First, a numerical solution is described of the equations in their present form (including the inertia term). Secondly, an analytical solution is described for the special case in which the inertia term in the momentum equation, $\mathrm{d}/\mathrm{d}x\,(U^2\delta)$, can be neglected, i.e. for fluids that have large Prandtl numbers. These solutions are described below.

Before continuing, consider the convective heat transfer, $\dot{Q}$, to the wall which can be written analytically as

$$\dot{Q} = -k\int\limits_0^L \left(\frac{\partial T}{\partial y}\right)_w b\,\mathrm{d}x = 2kb\Delta T_w \int\limits_0^L \frac{\mathrm{d}x}{\delta} \qquad (12)$$

where equation (5b) was used. The Nusselt number based on the plate half length, L, can be expressed as

$$Nu = \frac{\dot{Q}L}{k\Delta T_w bL} = 0{\cdot}467(Gr\cdot Pr)^{1/4}\int\limits_0^1 \frac{\mathrm{d}\bar{x}}{\bar{\delta}} \qquad (13)$$

where Gr is the dimensionless Grashof number that is defined as

$$Gr = L^3 g\beta\Delta T_w/\nu^2.$$

Fluids with arbitrary Prandtl number—numerical solution

The solution of equations (11) was accomplished through the use of a double precision Adams–Moulton numerical integration method. The results of the calculations to determine boundary layer depth distributions and depths at the plate center are given in Figs. 2 and 3. Figure 4 gives the heat transfer in terms of the Nusselt number, equation (13), as a function of the Prandtl number. The results of Fig. 3 show that conditions of $GrPr^2 > 5 \times 10^5\bar{\delta}_0^5$ will produce boundary-layer depths $(\delta/L)^2 < 0{\cdot}1$ which is consistent with the order-of-magnitude analysis in Appendix B.

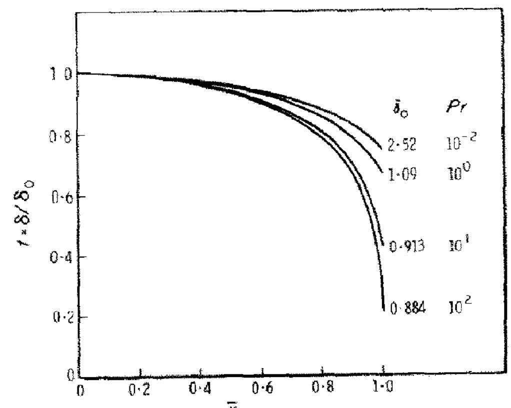

FIG. 2. Typical boundary layer depth distributions – including inertia terms.

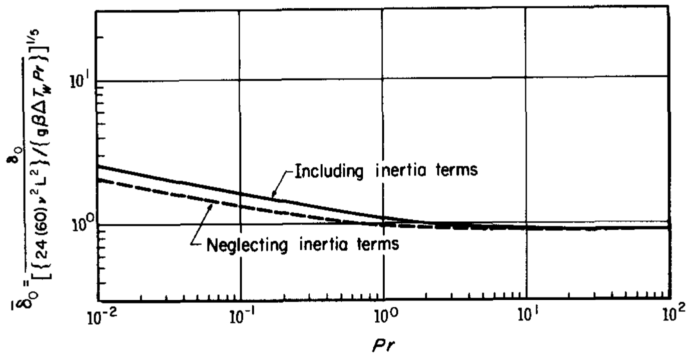

FIG. 3. Boundary layer depth at the plate center.

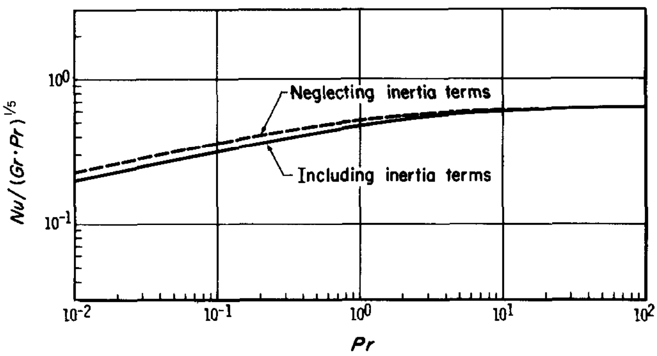

FIG. 4. Heat transfer.

Fluids with large Prandtl number

If the inertia term in equation (11a), $(d/dx)(U^2\delta)$, is neglected an analytical solution can be obtained. The justification of this simplification will be apparent upon comparison of the analytic results with the results of the numerical analysis just presented. Neglecting, then, the inertia term in equation (11a), equations (11a) and (11b) can be combined to yield

$$\bar{\delta}\frac{d}{d\bar{x}}\left(\frac{d\bar{\delta}^4}{d\bar{x}}\right) = -1. \qquad (14)$$

Now with

$$p = \frac{d\bar{\delta}^4}{d\bar{x}}, \quad \text{and} \quad \frac{dp}{d\bar{x}} = p\frac{dp}{d\bar{\delta}^4},$$

equation (14) can be integrated to yield

$$\frac{p^2}{2} = \frac{1}{2}\left(\frac{d\bar{\delta}^4}{d\bar{x}}\right)^2 = \frac{-4}{3}\bar{\delta}^3 + c. \qquad (15)$$

In order to satisfy the condition that $(d\bar{\delta}/d\bar{x})(o) = 0$, it will be necessary at this point to assume that the boundary layer depth at the plate center is known, i.e. $\bar{\delta}(0) = \bar{\delta}_0$. Later, the value of $\bar{\delta}_0$ that is consistent with the critical depth at the plate's edge, $\bar{\delta}_c$, will be obtained. Equation (15) is now expressed as

$$\frac{d\bar{\delta}}{d\bar{x}} = \frac{-1}{\sqrt{6}}\frac{1}{\bar{\delta}^3}(\bar{\delta}_0^3 - \bar{\delta}^3)^{\frac{1}{2}} \qquad (16)$$

where the negative sign has been retained rather than the positive sign since $d\bar{\delta}/d\bar{x}$ must be negative. Equation (16) is now integrated to yield

$$\int_{\bar{\delta}_0}^{\bar{\delta}} \frac{\bar{\delta}^3\,\mathrm{d}\bar{\delta}}{(\bar{\delta}_0^3 - \bar{\delta}^3)^{\frac{1}{2}}} = \frac{-\bar{x}}{\sqrt{6}} \tag{17}$$

or

$$\tfrac{2}{5}\bar{\delta}_0^{\frac{5}{2}}[t(1 - t^3)^{\frac{1}{2}} + 3^{-0.25}F(\beta, K)] = \bar{x}/\sqrt{6}$$

where $t = \bar{\delta}/\bar{\delta}_0$ and $F(\beta, K)$ is an elliptic integral of the first kind, i.e.

$$F(\beta, K) = \int_0^{\beta} \frac{\mathrm{d}\phi}{(1 - K^2 \sin^2 \phi)^{\frac{1}{2}}}$$

and the arguments are

$$\beta = \cos^{-1}\left(\frac{3^{\frac{1}{2}} - 1 + t}{3^{\frac{1}{2}} + 1 - t}\right), \quad K = \sin(75°)$$

The most simple procedure for evaluating equation (17) subject to the boundary condition (11d) is to (1) select a value of $\bar{\delta}_0$, (2) compute the depth distribution, $t(\bar{x})$, and finally (3) determine, with the edge depth, $\bar{\delta}_c$, and slope, $(\mathrm{d}\bar{\delta}/\mathrm{d}\bar{x})_c$, from equations (17) and (16), the value of the Prandtl number, Pr, from boundary condition (11d) that is necessary for the computed edge depth and slope to represent a critical condition at the plate's edge. Table 1 gives the results of calculations to determine the Prandtl number, Pr, and the boundary layer depth and slope at the plate's edge, $\bar{\delta}_c$ and $(\mathrm{d}\bar{\delta}/\mathrm{d}\bar{x})_c$, for various boundary layer depths at the plate center, $\bar{\delta}_0$. Figure 3 shows the boundary layer depth at the plate center as a function of Prandtl number and also gives a comparison with the numerically evaluated depth that includes the effects of the inertia term.

Table 1. Depth, $\bar{\delta}_c$, and slope, $(\mathrm{d}\bar{\delta}/\mathrm{d}\bar{x})_c$, at the plate's edge

$\bar{\delta}_0$	Pr	$\bar{\delta}_c$	$(\mathrm{d}\bar{\delta}/\mathrm{d}\bar{x})_c$
0·885	14·2	0·358	7·18
0·890	7·64	0·433	3·96
0·900	4·175	0·520	2·226
0·950	1·290	0·721	0·759
1·000	0·715	0·836	0·451
1·200	0·185	1·134	0·1455
1·500	0·0526	1·475	0·0523
2·000	0·0119	1·992	0·01583
5·000	0·00012	4·999	0·0004

The Nusselt number [equation (13)] can be expressed as

$$Nu = -0{\cdot}467\,(Gr \cdot Pr)^{\frac{1}{5}}\,4\bar{\delta}_c^3 \left(\frac{\mathrm{d}\bar{\delta}}{\mathrm{d}\bar{x}}\right)_c \tag{18}$$

where equation (14) has been integrated with respect to $\bar{x}$ from $\bar{x} = 0$ to $\bar{x} = 1{\cdot}0$. With the depth, $\bar{\delta}_c$, and slope, $(\mathrm{d}\bar{\delta}/\mathrm{d}\bar{x})_c$, at the plate's edge given in Table 1, the Nusselt number has been computed and is given in Fig. 4 as a function of the Prandtl number. Figure 4 also gives a comparison of the analytic results with the numerical results obtained using the complete equations, (11a)–(11d).

The agreement between the results of the numerical analysis which includes the inertia term with the analysis that neglects this term is quite good, as shown by Figs. 3 and 4, especially for fluids that have Prandtl numbers greater than unity.

For fluids with Prandtl numbers near unity, the Nusselt number can be expressed approximately as (see Fig. 4)

$$Nu = 0{\cdot}44\,(Gr \cdot Pr)^{\frac{1}{5}}. \tag{19}$$

Figure 5 gives a comparison of the present theory, equation (19), with a correlation [4] of appropriate experimental measurements for air. It is seen that equation (19) agrees relatively well with the experimental correlation equation within the range of Gr for which the experimental correlation is valid, i.e. $10^5 < Gr < 10^7$.

CONCLUSION

A physically reasonable and relatively accurate theory for the steady-state, two-dimensional, natural convection boundary layer on a finite-size, isothermal, horizontal plate for the cases of either a cold plate facing upward or a hot plate facing downward has been described. An essential point in this theory is the use of the condition that the boundary layer depth at the

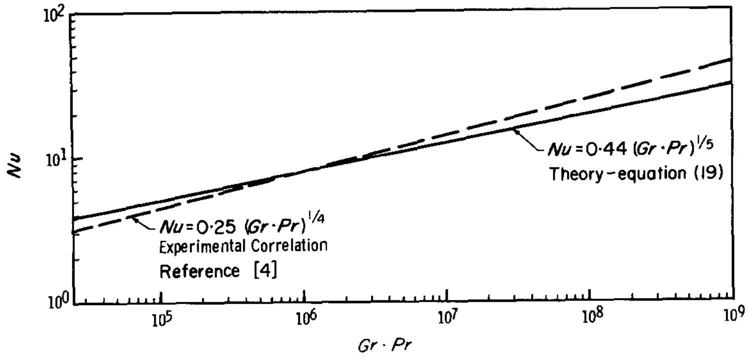

FIG. 5. Comparison of theory with experimental correlation equation for air.

plate edge is equal to the critical depth that is described in Appendix A.

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APPENDIX A

Boundary Layer Depth at the Plate Edge

The analysis of many boundary layer problems is considerably simplified by the condition that the boundary layer depth is zero at some conveniently defined point, such as the leading edge of the plate. This simplification is not possible in the analysis of the free-convection boundary layer which flows, under the influence of a hydrostatic pressure gradient, across a finite-size, horizontal plate i.e. the boundary layer depth is greater than zero over the entire plate (see Fig. 1). Therefore, the objective of this appendix is to define a characteristic depth for the natural-convection layer on a finite-size, horizontal plate. In order to accomplish this objective, the principle of a minimum or critical depth which is a well established part of the theory of open channel hydraulics, will be employed [9, 10]. Although, in the past this principle has been applied exclusively to the determination of the critical depth of liquids, with a free surface, flowing in open channels under the influence of a hydrostatic pressure gradient, a plausible extension may be made to the case of the free convection boundary layer flowing over a horizontal plate under a density stratified hydrostatic pressure gradient. This extension is certainly consistent with the assumption that a boundary layer approximation may be utilized in such a problem. In this Appendix, this extension is made, following the arguments of Bakhmeteff [9] to establish the boundary layer depth at the plate's edge.

The desired characteristic depth can be defined by examining the flow of mechanical energy within the boundary layer in connection with the postulate that there is a natural tendency for the denser fluid to seek a minimum depth consistent with the external conditions of the problem. In general the steady state conservation of mechanical energy of an incompressible fluid flowing through a stationary control volume can be expressed as [11]

$$\int_S \left(\frac{\bar{V}^2}{2} + gy + \frac{p}{\rho}\right) \rho \bar{V}\cdot d\bar{S} = -\int_S (\bar{V}\cdot\tau)\cdot d\bar{S} - \int_R (\tau\cdot\nabla)\cdot\bar{V} dR \qquad (A.1)$$

where $\bar{V}$, $d\bar{S}$ and τ represent the velocity vector, differential area vector on the surface of the control volume, R, and viscous stress tensor respectively. The first term on the right hand side of equation (A.1) represents the work, W, done by viscous forces to push the fluid into or out of the system, while the second term, Φ, represents the viscous dissipation of energy. Recall that equation (A.1) is a result of performing the scalar product of the velocity vector, $\bar{V}$, and the general momentum equation in vector form and is therefore strictly a mechanical energy equation; thermodynamic concepts have not been included in this equation. This equation must be satisfied independent of thermal energy considerations; it is coupled with the general energy equation only if the density and viscosity are temperature dependent.

Applied to a control volume enclosing a thin (Appendix B) free-convection boundary layer on a horizontal, finite-size plate, equation (A.1) becomes

$$\dot{E} = \dot{E}_0 - (W + \Phi), \qquad (A.2)$$

where

$$\dot{E} = \int_0^\delta \left(\frac{u^2}{2} + gy + p/\rho\right) \rho u b dy, \qquad (A.3)$$

and

$$\dot{E}_0 = \int_0^x \left(\frac{v^2}{2} + g\delta_0 + p_0/\rho\right) \rho v b dx. \qquad (A.4)$$

Equation (A.2) states simply that the rate at which mechanical energy flows in the boundary layer, $\dot{E}$, at a given position, x, along the plate is equal to the rate at which mechanical energy flows into the boundary layer through the exposed edge of the boundary layer, $\dot{E}_0$, minus the energy lost due to viscous work and viscous dissipation. For a thin boundary layer, the kinetic energy term in $\dot{E}_0$ can be neglected since $v^2 \ll u^2$, and the mechanical energy equation can be expressed per unit mass as

$$\dot{E}/\dot{M} = e = e_0 - (W + \Phi)/\dot{M} \qquad (A.5)$$

where $e_0 = p_0/\rho + g\delta_0$ is constant and $\dot{M}$ is the mass flow rate in the boundary layer defined by equation (9).

Now examine the average specific mechanical energy flowing at a given position x in the boundary layer, $\dot{E}/\dot{M} = e$. The hydrostatic pressure can be defined as (see Fig. A.1)

$$p = p_0 + \rho_0 g(\delta_0 - \delta) + \int_y^\delta \rho g dy \qquad (A.6)$$

and for small temperature variations the fluid density can be expressed as

$$\rho = \rho_0[1 - \beta(T - T_0)]. \qquad (A.7)$$

Further, the velocity and temperature distributions within the boundary layer are assumed to be given by

$$u/U = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \qquad T - T_0 = \Delta T_w\left(1 - \frac{y}{\delta}\right)^2. \qquad (A.8)$$

If equations (A.6)–(A.8) are substituted in equation (A.3) and the indicated integration performed, the average specific mechanical energy flowing past a given position x in the boundary layer can be expressed as

$$(e - g\delta_0 - p_0/\rho)\frac{21}{2\beta g \Delta T_w} = \bar{e} = \frac{54}{5g\beta\Delta T_w}\frac{\dot{M}^2}{\rho^2 b^2 \delta^2} + \delta. \qquad (A.9)$$

The variation of specific mechanical energy e or $\bar{e}$ with δ is illustrated in Fig. A.1. For a given value of mass flow rate, $\dot{M}$, Fig. A.1 shows that as the fluid moves from section (a) to (b) the loss in mechanical energy, $\bar{e}$, due to viscous dissipation and viscous work, corresponds to a decrease in the boundary layer depth. Note in Fig. A.1 that the natural tendency of the liquid

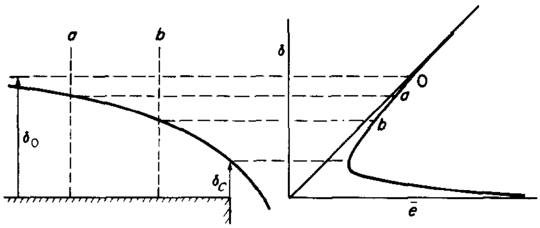

FIG. A.1. Variation of boundary layer depth with specific mechanical energy.

depth to decrease is limited to a minimum value, known as the critical depth, δ_c. Any further reduction in depth below δ_c would be possible only if external mechanical energy were added to the boundary layer. Therefore, the critical depth is the lowest depth to which the boundary layer may drop in the natural process of dissipating energy.

In order to select a preferred depth from the infinite number of possible depths shown in Fig. A.1 it is necessary to use the postulate that there is a natural tendency for the denser fluid within the boundary layer to seek a *minimum* depth consistent with the external conditions of the problem, i.e. the denser fluid tends to flow off the plate. With this postulate, and because of the continuous reduction in depth in the flow direction, the minimum or critical depth establishes itself at the plate edge. The upstream depth distribution will adjust itself in a manner consistent with the critical depth at the plate edge and the external conditions of the problem.

Note that the critical depth corresponds to the condition of minimum specific mechanical energy at the plate edge. Thus the critical depth can be evaluated by setting the derivative of e or $\bar{e}$ [equation (A.9)] with respect to δ equal to zero, and solving the resulting expression for δ_c, thus

$$\delta_c = \left(\frac{108}{5}\frac{\dot{M}_c^2}{\rho^2 b^2 g\beta\Delta T_w}\right)^{\frac{1}{3}} \tag{A.10}$$

where $\dot{M}_c$ represents the mass flow rate within the boundary layer at the plate edge.

Beij [12] has shown experimentally that the above argument is applicable to a variable mass flow rate across the plate providing $\dot{M}_c$ in equation (A.10) represents the mass flow rate at the plate edge. An extensive experimental study by Beij shows excellent agreement between measured and predicted (using the above critical depth principle) liquid depths at the free end of horizontal channels for both constant and spatially variable steady flow.

Therefore, the minimum or critical depth at the plate edge is considered to be defined by the expression

$$\left\{\frac{\partial}{\partial\delta}\int_0^\delta\left(\frac{u^2}{2} + gy + p/\rho\right)\rho u b\,\mathrm{d}y\right\}_{\dot{M}_c} = 0. \tag{A.11}$$

This expression is valid for the steady flow of thin fluid layers (including boundary layers in which the vertical component of velocity, v, is small compared with the horizontal component, u) across a horizontal plate under the influence of a hydrostatic pressure gradient and off the plate's free edge.

APPENDIX B

Order-of-Magnitude Analysis

The equations which describe the steady-state, two-dimensional, motion (natural-convection) of a constant property fluid across a finite-size, horizontal plate are

x-momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{B.1}$$

y-momentum

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g[1 - \beta(T - T_0)] \tag{B.2}$$

energy (neglecting viscous dissipation of energy)

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (B.3)$$

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (B.4)$$

The objective of this Appendix is to demonstrate the conditions under which the above equations can be simplified by neglecting terms that have a smaller order-of-magnitude effect on the results than that of the terms being retained.

If the variables u, v, T, x and y are made nondimensional and of order unity by appropriate reference velocities, temperatures and lengths, the nondimensional derivatives will also be of order unity and therefore the influence of each term in the above equations will depend on the magnitude of its coefficients. The nondimensional variables are defined as

$$\bar{x} = \frac{x}{L}, \bar{y} = \frac{y}{\delta_0}, \bar{u} = \frac{u}{u_r}, \bar{v} = \frac{v}{v_r},$$

$$\phi = \frac{T - T_0}{T_w - T_0} = \frac{T - T_0}{\Delta T_w}.$$

The quantities $\bar{x}$, $\bar{y}$ and ϕ are obviously of order unity. Special consideration, however, must be given to the selection of representative reference velocities u_r and v_r so that $\bar{u}$ and $\bar{v}$ will also be of order unity.

Since the mass flow rate in the x-direction increases from a value of zero at the plate center to a maximum value at the plate edge (see Fig. 1) a representative horizontal reference velocity is the average horizontal velocity of the fluid at the plate edge. This horizontal reference velocity is easily obtained by applying a momentum balance in the x-direction to the fluid located above the plate and between vertical boundaries at $x = 0$ and $x = L$. If viscous forces are neglected and the net driving force is equal to the integral of the excess hydrostatic pressure along the vertical boundary at $x = 0$ due to the denser fluid near the plate, an upper limit on the average horizontal velocity at the plate edge can be expressed as

$$u_r \cong \sqrt{(g\beta\Delta T_w\delta_0/10)}. \qquad (B.5)$$

A mass balance applied to the control volume shown in Fig. 1 yields the following representative vertical reference velocity

$$v_r = u_r\delta_0/L. \qquad (B.6)$$

With the nondimensional variables defined in equations (B.5) and (B.6), the nondimensional form of equations (B.1)–(B.4) are

x-momentum

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho u_r^2}\frac{\partial p}{\partial \bar{x}} + \left(\frac{10}{Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{5}{2}}\left[\left(\frac{\delta_0}{L}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right] \qquad (B.7)$$

y-momentum

$$\left(\frac{\delta_0}{L}\right)^2\left[\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right] = -\frac{1}{\rho u_r^2}\frac{\partial p}{\partial \bar{y}} + \left(\frac{10}{Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{5}{2}}\left(\frac{\delta_0}{L}\right)^2\left[\left(\frac{\delta_0}{L}\right)^2\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right] + $$

$$-\frac{10}{\beta\Delta T_w}(1 - \beta\Delta T_w\phi) \qquad (B.8)$$

energy

$$\bar{u}\frac{\partial \phi}{\partial \bar{x}} + \bar{v}\frac{\partial \phi}{\partial \bar{y}} = \left(\frac{10}{Pr^2 Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{5}{2}}\left[\left(\frac{\delta_0}{L}\right)^2\frac{\partial^2 \phi}{\partial \bar{x}^2} + \frac{\partial^2 \phi}{\partial \bar{y}^2}\right] \qquad (B.9)$$

continuity

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \qquad (B.10)$$

In the above nondimensional equations all dependent variables and their derivatives are of order unity, and the influence of each term depends on the magnitude of its coefficient. If $(\delta_0/L)^2 < 0{\cdot}1$, and the terms that have an order-of-magnitude of 0·1 or less are neglected the above equations can be simplified to the form *x-momentum*

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho u_r^2}\frac{\partial p}{\partial \bar{x}} + \left(\frac{10}{Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{1}{2}}\frac{\partial^2 \bar{u}}{\partial \bar{o}^2} \quad \text{(B.11)}$$

y-momentum

$$\frac{1}{\rho u_r^2}\frac{\partial p}{\partial \bar{y}} = -\frac{10}{\beta \Delta T_w}(1 - \beta \Delta T_w \phi) \quad \text{(B.12)}$$

energy

$$\bar{u}\frac{\partial \phi}{\partial \bar{x}} + \bar{v}\frac{\partial \phi}{\partial \bar{y}} = \left(\frac{10}{Pr^2 Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{1}{2}}\frac{\partial^2 \phi}{\partial \bar{y}^2} \quad \text{(B.13)}$$

continuity

$$\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} = 0. \quad \text{(B.14)}$$

The simplified x- and y-momentum equations can be combined to yield

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = 10\frac{\partial}{\partial \bar{x}}\int_0^y \phi \, d\bar{y} + \left(\frac{10}{Gr}\right)^{\frac{1}{2}}\left(\frac{L}{\delta_0}\right)^{\frac{1}{2}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}. \quad \text{(B.15)}$$

Note that the first terms in equation (B.15) are of order unity or greater. Since the viscous term in the momentum equation and the conduction term in the energy equation must be of order unity near the wall, the coefficients

$$\frac{\delta_0}{L}\left(\frac{Gr}{10}\right)^{\frac{1}{2}} \text{ and } \frac{\delta_0}{L}\left(\frac{Pr^2 Gr}{10}\right)^{\frac{1}{2}}$$

must also be of order unity. Therefore if $(\delta_0/L)^2 < 0{\cdot}1$, the parameters Gr and $GrPr^2$ must be of order 10^5 or larger.

These restrictions (Gr and $GrPr^2 > 10^5$) represent a lower limit on the conditions under which equations (B.13)–(B.15) adequately describe the behavior of the free-convection boundary layer on a horizontal plate. Equations (B.13)–(B.15) will more accurately describe the boundary-layer behavior as Gr and $GrPr^2 \to \infty$ and the corresponding boundary-layer depth, $\delta_0/L \to 0$.

Résumé—Le problème de la convection naturelle bidimensionnelle en régime permanent sur une plaque horizontale isotherme de taille finie est examiné théoriquement dans le cas où une plaque regarde vers le haut ou bien une plaque chaude regarde vers le bas. Les équations de la couche limite pour la continuité, l'énergie et la quantité de mouvement sont résolues en employant une analyse intégrale afin de déterminer le transport de chaleur à la surface de la plaque. Une part essentielle de cette analyse est l'emploi du concept ou de la condition que la profondeur de la couche limite au bord de la plaque est égale à une profondeur critique. Les résultats de cette analyse sont en très bon accord avec l'équation de corrélation expérimentale existante pour l'air.

NATÜRLICHE KONVEKTION AN EINER ENDLICHEN WAAGERECHTEN PLATTE

Zusammenfassung—Das Problem der zweidimensionalen, stationären, freien Konvektion an einer endlichen, isothermen, waagerechten Platte wird theoretisch geprüft für den Fall, dass eine kalte Seite nach oben oder eine heisse Seite nach unten gerichtet ist. Die Grenzschichtgleichungen für Kontinuität, Energie und Impuls werden mit Hilfe einer Integralanalyse gelöst um den Wärmeübergang an der Plattenoberfläche zu bestimmen. Ein wesentlicher Teil dieser Analyse besteht in der Bedingung dass die Grenzschichttiefe am Plattenrand gleich einer kritischen Tiefe ist. Die Ergebnisse dieser Analyse stimmen sehr gut mit bestehenden experimentellen Korrelationsgleichungen für Luft überein.

Аннотация—Задача двумерной стационарной естественной конвекции изучается теоретически для случая горизонтальной изотермической холодной пластины конечных размеров, расположенной вверху, и аналогичной горячей пластины, расположенной внизу. Теплообмен на поверхности пластины определялся путем решения для пограничного слоя методами интегральных преобразований уравнений неразрывности, энергии и количества движения. Эта задача решена при том существенном условии, что толщина пограничного слоя на краю пластины равна критической. Результаты анализа очень хорошо согласуются с имеющимися экспериментальными обобщенными уравнениями для воздуха.